

Structure of states for which each localized dynamics reduces to a localized subdynamics

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We consider a bipartite quantum system S (including parties A and B), interacting with an environment E through a *localized* quantum dynamics \mathcal{F}_{SE} . We call a quantum dynamics \mathcal{F}_{SE} localized if, e.g., the party A is isolated from the environment and only B interacts with the environment: $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$, where id_A is the identity map on the part A and \mathcal{F}_{BE} is a completely positive (CP) map on the both B and E . We will show that the reduced dynamics of the system is also localized as $\mathcal{E}_S = id_A \otimes \bar{\mathcal{E}}_B$, where $\bar{\mathcal{E}}_B$ is a CP map on B , if and only if the initial state of the system-environment is a *Markov state*. We then generalize this result to the two following cases: when both A and B interact with a same environment, and when each party interacts with its local environment.

I. INTRODUCTION

Consider a quantum system B which undergoes the evolution given by a map $\bar{\mathcal{E}}_B$. It is usually argued that $\bar{\mathcal{E}}_B$ must be a completely positive (CP) map [1]. This is so because we can always consider another quantum system A which is remained unchanged during the evolution of B . So the evolution of the combined system $S = AB$, for arbitrary state $\rho_S = \rho_{AB}$, is given by $id_A \otimes \bar{\mathcal{E}}_B$, where id_A is the identity map on the part A . $id_A \otimes \bar{\mathcal{E}}_B$ must be a positive map (i.e. it must map each positive operator to a positive operator), which means that $\bar{\mathcal{E}}_B$ must be a CP map.

When the system A is remained unchanged, its evolution, obviously, can be represented by id_A . In the above argument, it is assumed that, in addition, when the evolution of B is given by $\bar{\mathcal{E}}_B$, then the evolution of the combined system $S = AB$ is as $id_A \otimes \bar{\mathcal{E}}_B$. But, as first remarked by Pechukas [2], there is no reason that this will be the case, in general.

In this paper, we consider the case that only the part B of our bipartite system $S = AB$ interacts with an environment E . The evolution of the whole SE is given by $id_A \otimes \mathcal{F}_{BE}$, where \mathcal{F}_{BE} is a CP map on the both B and E . So, the reduced state of A remains unchanged during the evolution and the reduced dynamics of A can be represented by the identity map id_A . Now, we question whether the reduced dynamics of $S = AB$ can be represented as $id_A \otimes \bar{\mathcal{E}}_B$, where $\bar{\mathcal{E}}_B$ is a CP map.

Using the results of [3, 4], in the next section, we will see that each *localized* dynamics as $id_A \otimes \mathcal{F}_{BE}$, for the whole SE , reduces to a localized subdynamics as $id_A \otimes \bar{\mathcal{E}}_B$, if and only if the initial state of SE be a so-called *Markov state*.

Therefore, if the initial state of SE is not a Markov state, there is no guarantee that a dynamics as $id_A \otimes \mathcal{F}_{BE}$ reduces to a subdynamics as $id_A \otimes \bar{\mathcal{E}}_B$. In fact, one can find explicit examples for which localized dynamics do not reduce to localized subdynamics. In other words, one can find explicit examples for which, though the reduced dynamics of A is given by id_A , but the reduced dynamics of AB can not be represented as $id_A \otimes \bar{\mathcal{E}}_B$. Such kind of examples will be given in the next section.

In our discussion in Sec. II, we use theorem 1 of [4]. During the proof of theorem 1 in Ref. [4], it is assumed that the final Hilbert spaces, after the evolution, can differ from the initial ones. Whether this assumption can be relaxed, is discussed in Sec. III.

In Secs. IV and V, we come back to our main subject and generalize the result of Sec. II. In Sec. IV, we consider the case that the both parts of the system, A and B , can interact with a same environment and, in Sec. V, we consider the case that each part of the system interacts with its local environment. We end our paper in Sec. VI, with a brief review of our results.

II. STRUCTURE OF INITIAL ρ_{SE} FOR WHICH LOCALIZED DYNAMICS REDUCES TO LOCALIZED SUBDYNAMICS

The quantum dynamics of a finite dimensional system can be written as

$$\rho \rightarrow \rho' = \mathcal{F}(\rho) \equiv \sum_j F_j \rho F_j^\dagger, \quad \sum_j F_j^\dagger F_j = I, \quad (1)$$

where ρ and ρ' are the initial and final states (density operators) of the system, respectively. $\{F_j\}$ is a set of linear operators on \mathcal{H} (\mathcal{H} is the Hilbert space of the system) and I is the identity operator on \mathcal{H} [1]. Such kind of evolution, given by Eq. (1), is called completely positive (CP) evolution [1, 5]. In addition, if the summation in Eq. (1) includes only one term, with $F_1 = U$, for a

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unitary U , then the evolution is called a unitary time evolution; otherwise, the evolution is called a (generalized) measurement.

Assume that the whole system-environment undergoes a CP evolution as Eq. (1). In addition, consider the case that the system itself is bipartite $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$ and the evolution \mathcal{F} in Eq. (1) is as $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$, where id_A is the identity map on $\mathcal{L}(\mathcal{H}_A)$ and \mathcal{F}_{BE} is a CP map on $\mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_E)$ ($\mathcal{L}(\mathcal{H})$ is the space of linear operators on the Hilbert space \mathcal{H}). So the linear operators F_j in Eq. (1) are in the following form:

$$F_j = I_A \otimes f_j, \quad \sum_j f_j^\dagger f_j = I_{BE}, \quad (2)$$

where I_A (I_{BE}) is the identity operator on \mathcal{H}_A ($\mathcal{H}_B \otimes \mathcal{H}_E$) and f_j are linear operators acting on $\mathcal{H}_B \otimes \mathcal{H}_E$. We call such a map *localized* since it acts only on BE . In other words, the party B interacts with the environment E through \mathcal{F}_{BE} , but the party A is isolated from the environment and its state remains unchanged during the evolution \mathcal{F}_{SE} . Now a naturally arisen question is that whether the reduced dynamics of the system S is also localized.

To find the general structure of initial ρ_{ABE} for which any localized dynamics leads to a localized subdynamics, we first need to recall the definition of *Markov states* [3]. For an arbitrary tripartite state ρ_{ABE} , one can find a CP *assignment map* Λ such that $\rho_{ABE} = \Lambda(\rho_{AB})$, where $\rho_{AB} = \text{Tr}_E(\rho_{ABE})$. For example, Λ can be constructed as $\Lambda = \bar{\Lambda} \circ \Xi$. The CP map Ξ is defined as $\Xi(\rho_{AB}) = (I_{AB} \otimes |0_E\rangle)\rho_{AB}(I_{AB} \otimes \langle 0_E|)$, where $|0_E\rangle$ is a fixed state in \mathcal{H}_E . The completely positive map $\bar{\Lambda}$, which maps $\rho_{AB} \otimes |0_E\rangle\langle 0_E|$ to the ρ_{ABE} , can be found, e.g., using the method introduced in Ref. [6]. In fact there are infinite number of CP assignment maps Λ which map ρ_{AB} to ρ_{ABE} . However, if one can find a CP assignment map Λ as

$$\Lambda = id_A \otimes \Lambda_B, \quad (3)$$

i.e. if $\rho_{ABE} = id_A \otimes \Lambda_B(\rho_{AB})$, where $\Lambda_B : \mathcal{L}(\mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_E)$ is a CP assignment map on $\mathcal{L}(\mathcal{H}_B)$, then the tripartite state ρ_{ABE} is called a Markov state [3].

For a Markov state ρ_{ABE} , it has been shown in Ref. [3] that there exists a decomposition of the Hilbert space \mathcal{H}_B as $\mathcal{H}_B = \bigoplus_k \mathcal{H}_{b_k^L} \otimes \mathcal{H}_{b_k^R}$ such that

$$\rho_{ABE} = \bigoplus_k q_k \rho_{Ab_k^L} \otimes \rho_{b_k^R E}, \quad (4)$$

where $\{q_k\}$ is a probability distribution, $\rho_{Ab_k^L}$ is a state on $\mathcal{H}_A \otimes \mathcal{H}_{b_k^L}$ and $\rho_{b_k^R E}$ is a state on $\mathcal{H}_{b_k^R} \otimes \mathcal{H}_E$. From Eq. (4), we see that $\rho_{AB} = \bigoplus_k q_k \rho_{Ab_k^L} \otimes \rho_{b_k^R}$, where $\rho_{b_k^R} = \text{Tr}_E(\rho_{b_k^R E})$. So, the assignment map Λ in Eq. (3) is as $\Lambda = \bigoplus_k id_{Ab_k^L} \otimes \Lambda_{b_k^R}$, where $id_{Ab_k^L}$ is the identity map on $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_{b_k^L})$ and $\Lambda_{b_k^R} : \mathcal{L}(\mathcal{H}_{b_k^R}) \rightarrow \mathcal{L}(\mathcal{H}_{b_k^R} \otimes \mathcal{H}_E)$ is a CP assignment map on $\mathcal{L}(\mathcal{H}_{b_k^R})$ such that $\Lambda_{b_k^R}(\rho_{b_k^R}) = \rho_{b_k^R E}$.

Using Eqs. (2) and (3), we have

$$\begin{aligned} \rho'_{AB} &= \text{Tr}_E \circ \mathcal{F}_{SE}(\rho_{ABE}) \\ &= \text{Tr}_E \circ [id_A \otimes \mathcal{F}_{BE}] \circ [id_A \otimes \Lambda_B](\rho_{AB}) \\ &= id_A \circ [\text{Tr}_E \circ \mathcal{F}_{BE} \circ \Lambda_B](\rho_{AB}) \\ &= id_A \circ \bar{\mathcal{E}}_B(\rho_{AB}), \end{aligned} \quad (5)$$

where $\bar{\mathcal{E}}_B \equiv \text{Tr}_E \circ \mathcal{F}_{BE} \circ \Lambda_B$ is a CP map on $\mathcal{L}(\mathcal{H}_B)$ (since it is a composition of three CP maps). In addition, $\rho_{AB} = \text{Tr}_E(\rho_{ABE})$ and ρ'_{AB} are the initial and final states of the system, respectively. So, when the initial ρ_{ABE} is a Markov state, then any arbitrary localized dynamics in Eq. (2), leads to a localized subdynamics as Eq. (5).

Interestingly, the reverse is also true: if for an initial state ρ_{ABE} , any localized dynamics in Eq. (2) leads to a localized subdynamics, then ρ_{ABE} is a Markov state. To prove this statement, we need a result of Ref. [4]. Assume that for a tripartite initial state ρ_{ABE} and an arbitrary localized $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$:

$$\begin{aligned} \rho'_{AB'E'} &= \sum_j (I_A \otimes f_j) \rho_{ABE} (I_A \otimes f_j^\dagger), \\ f_j : \mathcal{H}_B \otimes \mathcal{H}_E &\rightarrow \mathcal{H}_{B'} \otimes \mathcal{H}_{E'}, \quad \sum_j f_j^\dagger f_j = I_{BE}, \end{aligned} \quad (6)$$

we have

$$\begin{aligned} \rho'_{AB'} &= \sum_i (I_A \otimes \bar{E}_i) \rho_{AB} (I_A \otimes \bar{E}_i^\dagger), \\ \bar{E}_i : \mathcal{H}_B &\rightarrow \mathcal{H}_{B'}, \quad \sum_i \bar{E}_i^\dagger \bar{E}_i = I_B, \end{aligned} \quad (7)$$

where $\rho_{AB} = \text{Tr}_E(\rho_{ABE})$ is the initial state of the system and $\rho'_{AB'} = \text{Tr}_{E'}(\rho'_{AB'E'})$ is the final state of the system. In Eqs. (6) and (7), we assume that, in general, the final Hilbert spaces of the part B and the environment may differ from the initial ones.

The *mutual information* of a bipartite state ρ_{AB} is defined as $I(A : B)_\rho = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, where $\rho_A = \text{Tr}_B(\rho_{AB})$ and $\rho_B = \text{Tr}_A(\rho_{AB})$ are the reduced states and $S(\rho)$ is the von Neumann entropy of the state ρ : $S(\rho) = -\text{Tr}(\rho \log \rho)$ [1]. The mutual information $I(A : B')_{\rho'}$ of the final state $\rho'_{AB'}$ is defined similarly. Now, in the theorem 11.15 of Ref. [1], it has been shown that if Eq. (7) holds, then

$$I(A : B)_\rho \geq I(A : B')_{\rho'}. \quad (8)$$

Since we assume that an arbitrary localized $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$ in Eq. (6) leads to Eq. (7), so for an arbitrary localized $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$ in Eq. (6), Eq. (8) also holds. Theorem 1 of Ref. [4] states that if for any localized $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$ in Eq. (6), the inequality (8) holds, then the initial ρ_{ABE} is a Markov state. So assuming that, for arbitrary localized \mathcal{F}_{SE} , Eq. (6) leads to Eq. (7), results that the initial ρ_{ABE} is a Markov state.

In summary, the initial ρ_{ABE} is a Markov state, if and only if, Eq. (6) leads to Eq. (7), for arbitrary localized $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$.

Note that, in Eqs. (6) and (7), we assume that the final Hilbert spaces of the part B , $\mathcal{H}_{B'}$, and the environment, $\mathcal{H}_{E'}$, may be different from the initial \mathcal{H}_B and \mathcal{H}_E , respectively. In fact, during the proof of the theorem 1 in Ref. [4], this assumption has been used. Whether this assumption can be relaxed, is discussed in the next section.

We end this section with some illustrating examples.

Example 1. Consider the set $\mathcal{S} = \{\rho_{SE} = \rho_S \otimes \tilde{\omega}_E\}$, where ρ_S are arbitrary states of the system, but $\tilde{\omega}_E$ is a fixed state of environment. As it is famous [1], when the initial state of the system-environment is a member of \mathcal{S} and the whole system-environment undergoes a CP evolution as Eq. (1), then the reduced dynamics of the system is also CP.

Now, using Eq. (4), it can be shown easily that the factorized initial state $\rho_{ABE} = \rho_{AB} \otimes \tilde{\omega}_E$ is a Markov state. It is due to the case that $\mathcal{H}_B = \mathcal{H}_{b^L} \otimes \mathcal{H}_{b^R}$, where \mathcal{H}_{b^R} is a trivial one dimensional Hilbert space. So, for the factorized initial state, any localized dynamics in Eq. (6) leads to a localized subdynamics as Eq. (7).

Example 2. Consider the set

$$\mathcal{S} = \{\rho_{SE} = \sum_i p_i |\tilde{i}_S\rangle\langle\tilde{i}_S| \otimes \tilde{\omega}_i\}, \quad (9)$$

where $\{p_i\}$ is arbitrary probability distribution ($p_i \geq 0$, $\sum_i p_i = 1$), but $\{|\tilde{i}_S\rangle\}$ is a fixed orthonormal basis for \mathcal{H}_S and $\tilde{\omega}_i$ are fixed density operators on \mathcal{H}_E . It has been shown in [7] that when the initial state of the system-environment is a member of \mathcal{S} , then, similar to the factorized initial states in the previous example, the reduced dynamics of the system is CP (for any CP evolution of the whole system-environment).

But, when the initial ρ_{ABE} is a member of the set given in Eq. (9), then comparing Eqs. (9) and (4) shows that, in general, ρ_{ABE} is not a Markov state. For example, consider the case that $\mathcal{H}_E = \bigoplus_i \mathcal{H}_{E_i}$ and $\tilde{\omega}_i$ in Eq. (9) is a state on \mathcal{H}_{E_i} . Let's denote the projector onto the \mathcal{H}_{E_i} by Π_{E_i} . So, from Eq. (9), we have

$$\text{Tr}_E(I_{AB} \otimes \Pi_{E_i} \rho_{ABE} I_{AB} \otimes \Pi_{E_i}) = p_i |\tilde{i}_{AB}\rangle\langle\tilde{i}_{AB}|.$$

If ρ_{ABE} can be written as Eq. (4) too, then

$$\begin{aligned} & \text{Tr}_E(I_{AB} \otimes \Pi_{E_i} \rho_{ABE} I_{AB} \otimes \Pi_{E_i}) \\ &= \bigoplus_k q_k \rho_{Ab_k^L} \otimes \text{Tr}_E(I_{b_k^R} \otimes \Pi_{E_i} \rho_{b_k^R E} I_{b_k^R} \otimes \Pi_{E_i}) \\ &= \bigoplus_k q_k \rho_{Ab_k^L} \otimes q^{(ik)} \rho_{b_k^R}^{(i)}, \end{aligned}$$

where $0 \leq q^{(ik)} \leq 1$ and $\rho_{b_k^R}^{(i)}$ is a state on $\mathcal{H}_{b_k^R}$ such that $q^{(ik)} \rho_{b_k^R}^{(i)} = \text{Tr}_E(I_{b_k^R} \otimes \Pi_{E_i} \rho_{b_k^R E} I_{b_k^R} \otimes \Pi_{E_i})$. So

$$\bigoplus_k q_k \rho_{Ab_k^L} \otimes q^{(ik)} \rho_{b_k^R}^{(i)} = p_i |\tilde{i}_{AB}\rangle\langle\tilde{i}_{AB}|.$$

Note that $\rho_{Ab_k^L} \otimes \rho_{b_k^R}^{(i)}$ belong to different subspaces $\mathcal{H}_A \otimes \mathcal{H}_{B_k}$ ($\mathcal{H}_{B_k} \equiv \mathcal{H}_{b_k^L} \otimes \mathcal{H}_{b_k^R}$). Therefore the above

equality holds with only one term in the summation, i.e. only one $q^{(ik)}$ is non-zero: $q_k \rho_{Ab_k^L} \otimes q^{(ik)} \rho_{b_k^R}^{(i)} = p_i |\tilde{i}_{AB}\rangle\langle\tilde{i}_{AB}|$. In addition, $\rho_{Ab_k^L}$ and $\rho_{b_k^R}^{(i)}$ must be pure states: $|\psi_{Ab_k^L}\rangle\langle\psi_{Ab_k^L}| \otimes |\varphi_{b_k^R}^{(i)}\rangle\langle\varphi_{b_k^R}^{(i)}| = |\tilde{i}_{AB}\rangle\langle\tilde{i}_{AB}|$. When $|\tilde{i}_{AB}\rangle$ is not a separable state as $|\psi_{Ab_k^L}\rangle \otimes |\varphi_{b_k^R}^{(i)}\rangle$, the above equality does not hold. So, in general, the initial ρ_{ABE} , chosen from the set in Eq. (9), can not be written as Eq. (4).

Therefore, for set given in Eq. (9), there is no guarantee that a localized dynamics as Eq. (6) reduces to a localized subdynamics as Eq. (7). In our forthcoming work, an explicit example for which a localized dynamics as Eq. (6) does not reduce to a localized subdynamics as Eq. (7) (for initial states which can be written as Eq. (9)) will be appeared.

Example 3. Consider the set

$$\begin{aligned} \mathcal{S} &= \{\rho_{SE} = \bigoplus_i p_i \rho_{L_i} \otimes \tilde{\omega}_{R_i E}\}, \\ \mathcal{H}_S &= \bigoplus_i \mathcal{H}_{L_i} \otimes \mathcal{H}_{R_i}, \end{aligned} \quad (10)$$

where $\{p_i\}$ is arbitrary probability distribution, ρ_{L_i} is arbitrary state on \mathcal{H}_{L_i} , but $\tilde{\omega}_{R_i E}$ is a fixed state on $\mathcal{H}_{R_i} \otimes \mathcal{H}_E$. This set of initial ρ_{SE} also leads to CP reduced dynamics, for arbitrary CP evolution for the whole system-environment [4, 8]. In fact, the set given in Eq. (10) is the most general possible set of initial ρ_{SE} which leads to CP reduced dynamics, if we restrict ourselves to the case of CP assignment map [8] (see also [9]).

When are all $\rho_{ABE} \in \mathcal{S}$ in Eq. (10) Markov states and so for them Eq. (6) leads to Eq. (7), for any arbitrary localized dynamics $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$?

From Eq. (10) we know that $\mathcal{H}_{AB} = \bigoplus_i \mathcal{H}_{L_i} \otimes \mathcal{H}_{R_i}$. In order that ρ_{ABE} be a Markov state, from Eq. (4), we see that, in addition, we must have $\mathcal{H}_B = \bigoplus_k \mathcal{H}_{b_k^L} \otimes \mathcal{H}_{b_k^R}$. So the simplest case occurs when $\mathcal{H}_{AB} = \bigoplus_i \mathcal{H}_A \otimes \mathcal{H}_{B_i} = \bigoplus_i (\mathcal{H}_A \otimes \mathcal{H}_{b_i^L}) \otimes \mathcal{H}_{b_i^R}$, i.e. $\mathcal{H}_{L_i} = \mathcal{H}_A \otimes \mathcal{H}_{b_i^L}$ and $\mathcal{H}_{R_i} = \mathcal{H}_{b_i^R}$. Now, from Eq. (10), we have

$$\rho_{ABE} = \bigoplus_i p_i \rho_{Ab_i^L} \otimes \tilde{\omega}_{b_i^R E}, \quad (11)$$

which is in the form of Eq. (4) with $\rho_{b_i^R E} = \tilde{\omega}_{b_i^R E}$ which are fixed for all $\rho_{ABE} \in \mathcal{S}$ in Eq. (10).

A more general case occurs when \mathcal{H}_A also decomposes as $\mathcal{H}_A = \bigoplus_i \mathcal{H}_{A_i} = \bigoplus_i \mathcal{H}_{a_i^L} \otimes \mathcal{H}_{a_i^R}$. So

$$\begin{aligned} \mathcal{H}_{AB} &= (\bigoplus_i \mathcal{H}_{A_i}) \otimes (\bigoplus_j \mathcal{H}_{B_j}) = \bigoplus_{ij} \mathcal{H}_{A_i} \otimes \mathcal{H}_{B_j} \\ &= \bigoplus_{ij} (\mathcal{H}_{a_i^L} \otimes \mathcal{H}_{b_j^L}) \otimes (\mathcal{H}_{a_i^R} \otimes \mathcal{H}_{b_j^R}) \\ &= \bigoplus_{ij} \mathcal{H}_{L_{ij}} \otimes \mathcal{H}_{R_{ij}}. \end{aligned}$$

Then, from Eq. (10), we have

$$\rho_{ABE} = \bigoplus_{ij} p_{ij} \rho_{a_i^L b_j^L} \otimes \tilde{\omega}_{a_i^R b_j^R E}, \quad (12)$$

where $\{p_{ij}\}$ is a probability distribution, $\rho_{a_i^L b_j^L}$ is a state on $\mathcal{H}_{a_i^L} \otimes \mathcal{H}_{b_j^L}$ and $\tilde{\omega}_{a_i^R b_j^R E}$ is a state on $\mathcal{H}_{a_i^R} \otimes \mathcal{H}_{b_j^R} \otimes \mathcal{H}_E$. $\tilde{\omega}_{a_i^R b_j^R E}$ are fixed for all $\rho_{ABE} \in \mathcal{S}$ in Eq. (10). Now if $\tilde{\omega}_{a_i^R b_j^R E}$ be as $\tilde{\omega}_{a_i^R}^{(j)} \otimes \tilde{\omega}_{b_j^R E}$ where $\tilde{\omega}_{a_i^R}^{(j)}$ is a state on $\mathcal{H}_{a_i^R}$ and $\tilde{\omega}_{b_j^R E}$ is a state on $\mathcal{H}_{b_j^R} \otimes \mathcal{H}_E$, then Eq. (12) becomes as Eq. (4). Therefore, for this case, all $\rho_{ABE} \in \mathcal{S}$ in Eq. (10) are Markov states and so any localized dynamics for them as Eq. (6) reduces to a localized subdynamics as Eq. (7).

Example 4. (example 2 of [10]) In example 2, we encountered a case for which $\rho_{ABE} \in \mathcal{S}$ in Eq. (9) are not Markov states. So, for them, one can find at least one localized dynamics as Eq. (6) which does not reduce to a localized subdynamics as Eq. (7). There, the initial states of the system $\rho_{AB} = \text{Tr}_E(\rho_{ABE})$ are in the restricted form $\sum_i p_i |\tilde{i}_{AB}\rangle \langle \tilde{i}_{AB}|$. So, $\rho_B = \text{Tr}_A(\rho_{AB}) = \sum_i p_i \tilde{\omega}_B^{(i)}$, where $\tilde{\omega}_B^{(i)} = \text{Tr}_A(|\tilde{i}_{AB}\rangle \langle \tilde{i}_{AB}|)$ are fixed states on \mathcal{H}_B , are also restricted. Here, we consider a case for which initial ρ_B are arbitrary.

Assume that the set of initial ρ_{ABE} is given by

$$\mathcal{S} = \{\rho_{ABE} = \rho_B \otimes \rho_{AE} : \text{Tr}_A(\rho_{AE}) = \tilde{\omega}\}, \quad (13)$$

where $\tilde{\omega}$ is a fixed state. Consider the case that the dynamics is localized as $id_A \otimes Ad_U$, where U is the swap operator $U|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle$ and $Ad_U(X) \equiv UXU^\dagger$, for arbitrary $X \in \mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_E)$ (note that U acts on $\mathcal{H}_B \otimes \mathcal{H}_E$).

In this example, the reduced dynamics of the part A is given by id_A . In addition, the reduced dynamics of the part B is given by the CP map $\bar{\mathcal{E}}_B(\rho_B) = \tilde{\omega}$. But, the reduced dynamics of $S = AB$ is not given by $id_A \otimes \bar{\mathcal{E}}_B$, in general. The final state of AB , after the evolution, is given by ρ_{AE} , which is the initial state of AE . But, we have

$$id_A \otimes \bar{\mathcal{E}}_B(\rho_{AB}) = id_A(\rho_A) \otimes \bar{\mathcal{E}}_B(\rho_B) = \rho_A \otimes \tilde{\omega},$$

where $\rho_A = \text{Tr}_E(\rho_{AE})$ and so $\rho_{AB} = \text{Tr}_E(\rho_B \otimes \rho_{AE}) = \rho_A \otimes \rho_B$. In general, ρ_{AE} differs from $\rho_A \otimes \tilde{\omega}$, so the reduced dynamics of AB is not given by $id_A \otimes \bar{\mathcal{E}}_B$.

When $\rho_{AE} = \rho_A \otimes \tilde{\omega}$, then the initial ρ_{ABE} is a Markov state as Eq. (4) and so the reduced dynamics of AB is given by $id_A \otimes \bar{\mathcal{E}}_B$. But, when $\rho_{AE} \neq \rho_A \otimes \tilde{\omega}$, then the initial ρ_{ABE} is not a Markov state and the reduced dynamics of AB is not given by $id_A \otimes \bar{\mathcal{E}}_B$.

III. THE INITIAL ρ_{ABE} IS A MARKOV STATE IFF EACH LOCALIZED DYNAMICS DIRECTLY REDUCES TO A LOCALIZED SUBDYNAMICS

In Sec. II, we have seen that if each localized dynamics as Eq. (6) leads to a localized subdynamics as Eq. (7),

then the initial ρ_{ABE} is a Markov state as Eq. (4) and vice versa. In Eqs. (6) and (7), the final $\mathcal{H}_{B'}$ and $\mathcal{H}_{E'}$ may differ from the initial \mathcal{H}_B and \mathcal{H}_E , respectively. There, we have questioned whether this condition can be relaxed. In this section, we discuss about this subject.

Consider a CP assignment map Λ which maps ρ_{AB} to ρ_{ABE} :

$$\begin{aligned} \rho_{ABE} &= \Lambda(\rho_{AB}) = \sum_l R_l \rho_{AB} R_l^\dagger, \\ R_l : \mathcal{H}_A \otimes \mathcal{H}_B &\rightarrow \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E, \\ \sum_l R_l^\dagger R_l &= I_{AB}. \end{aligned} \quad (14)$$

So, for a localized dynamics $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$ as

$$\begin{aligned} \rho'_{ABE} &= \sum_j (I_A \otimes f_j) \rho_{ABE} (I_A \otimes f_j^\dagger), \\ f_j : \mathcal{H}_B \otimes \mathcal{H}_E &\rightarrow \mathcal{H}_B \otimes \mathcal{H}_E, \\ \sum_j f_j^\dagger f_j &= I_{BE}, \end{aligned} \quad (15)$$

we have

$$\rho'_{ABE} = \sum_{jl} N_{jl} \rho_{AB} N_{jl}^\dagger,$$

where $N_{jl} \equiv I_A \otimes f_j R_l$ and $\sum_{jl} N_{jl}^\dagger N_{jl} = I_{AB}$. Therefore

$$\begin{aligned} \rho'_{AB} &= \sum_{jkl} \langle k_E | N_{jl} \rho_{AB} N_{jl}^\dagger | k_E \rangle \\ &= \sum_l X_{jkl} \rho_{AB} X_{jkl}^\dagger, \end{aligned} \quad (16)$$

where $\{|k_E\rangle\}$ is an orthonormal basis for \mathcal{H}_E , $X_{jkl} \equiv \langle k_E | N_{jl}$ is a linear operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\sum_l X_{jkl}^\dagger X_{jkl} = I_{AB}$. Now if

$$\begin{aligned} X_{jkl} &= I_A \otimes \bar{E}_{jkl}, \\ \bar{E}_{jkl} : \mathcal{H}_B &\rightarrow \mathcal{H}_B, \\ \sum_l \bar{E}_{jkl}^\dagger \bar{E}_{jkl} &= I_B, \end{aligned} \quad (17)$$

then

$$\begin{aligned} \rho'_{AB} &= \sum_{jkl} (I_A \otimes \bar{E}_{jkl}) \rho_{AB} (I_A \otimes \bar{E}_{jkl}^\dagger), \\ \bar{E}_{jkl} : \mathcal{H}_B &\rightarrow \mathcal{H}_B, \\ \sum_{jkl} \bar{E}_{jkl}^\dagger \bar{E}_{jkl} &= I_B; \end{aligned} \quad (18)$$

i.e. the reduced dynamics \mathcal{E}_{AB} is also localized: $\mathcal{E}_{AB} = id_A \otimes \bar{\mathcal{E}}_B$.

If Eq. (17) holds, we say that the localized dynamics $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$ directly reduces to the localized subdynamics $\mathcal{E}_{AB} = id_A \otimes \bar{\mathcal{E}}_B$. From Eq. (5), we know that,

for a Markov state ρ_{ABE} , there exists a CP assignment map as Eq. (3) such that each $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$ directly reduces to a $\mathcal{E}_{AB} = id_A \otimes \bar{\mathcal{E}}_B$. In the following, we show that the reverse is also true: if for a tripartite state ρ_{ABE} there exists a CP assignment map Λ such that each localized dynamics $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$ directly reduces to a localized sub-dynamics $\mathcal{E}_{AB} = id_A \otimes \bar{\mathcal{E}}_B$, then this Λ is as Eq. (3), i.e. ρ_{ABE} is a Markov state.

Assume that for a CP assignment map Λ as Eq. (14) and any arbitrary $\mathcal{F}_{SE} = id_A \otimes \mathcal{F}_{BE}$ in Eq. (15), Eq. (17) holds:

$$\langle k_E | I_A \otimes f_j R_l = I_A \otimes \bar{E}_{jkl}. \quad (19)$$

Note that each R_l in Eq. (14) can be decomposed as

$$R_l = \sum_m A_m \otimes B_{lm}, \quad (20)$$

where $A_m \in \mathcal{L}(\mathcal{H}_A)$ and $B_{lm} : \mathcal{H}_B \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$. The set $\mathcal{S}_A = \{A_m\}$ is chosen as each $A \in \mathcal{L}(\mathcal{H}_A)$ can be written as a linear decomposition of A_m . For example, when \mathcal{H}_A is two dimensional, \mathcal{S}_A can be chosen as $\mathcal{S}_A = \{I, \sigma_x, \sigma_y, \sigma_z\}$, where σ_x, σ_y and σ_z are the Pauli operators. Note that $A_m \in \{I, \sigma_x, \sigma_y, \sigma_z\}$ are both unitary and orthonormal as

$$\text{Tr}(A_m^\dagger A_n) = 2 \delta_{mn}.$$

This way of constructing \mathcal{S}_A can be generalized to the higher dimensional cases too. For each finite dimensional \mathcal{H}_A (with the dimension d_A), one can find a set $\mathcal{S}_A = \{A_m\}$, including $(d_A)^2$ unitary operators, as a basis for $\mathcal{L}(\mathcal{H}_A)$, where $A_1 = I_A$ and A_m are orthonormal as

$$\text{Tr}(A_m^\dagger A_n) = d_A \delta_{mn}; \quad (21)$$

see, e.g., the appendix B of Ref. [11].

Replacing (20) in (19), we get

$$\sum_m A_m \otimes \langle k_E | f_j B_{lm} = I_A \otimes \bar{E}_{jkl}.$$

So, using Eq. (21), we see that $\bar{E}_{jkl} = \langle k_E | f_j B_{l1}$ and, for $m \neq 1$, we have

$$\langle k_E | f_j B_{lm} = 0. \quad (22)$$

Choosing $\{|n_B\rangle\}$ as an orthonormal basis for \mathcal{H}_B , the above equation means that for $m \neq 1$:

$$\begin{aligned} \langle k_E | \langle n'_B | f_j B_{lm} | n_B \rangle &= 0 \\ \Rightarrow \langle \varphi_{jkn'} | \psi_{lmn} \rangle &= 0, \end{aligned}$$

where $|\psi_{lmn}\rangle \equiv B_{lm}|n_B\rangle$ and $|\varphi_{jkn'}\rangle \equiv f_j^\dagger |n'_B\rangle |k_E\rangle$. Now, if $|\psi_{lmn}\rangle$ be a non-zero state, then, since f_j is arbitrary, one can always find a $|\varphi_{jkn'}\rangle$ such that $\langle \varphi_{jkn'} | \psi_{lmn} \rangle \neq 0$. Therefore, for $m \neq 1$ (and all n), $|\psi_{lmn}\rangle = 0$; so, for $m \neq 1$, $B_{lm} = 0$. Finally, R_l in Eq. (20) is as

$$R_l = I_A \otimes B_{l1}, \quad (23)$$

which means that the CP assignment map Λ in Eq. (14) is as Eq. (3). So, ρ_{ABE} is a Markov state and, using the result of Ref. [3], can be written as Eq. (4).

In summary, we proved that the initial ρ_{ABE} is a Markov state if and only if there exists a CP assignment map such that each localized dynamics as Eq. (15) *directly* reduces to a localized subdynamics as Eq. (18).

The above discussion shows that when the initial ρ_{ABE} is not a Markov state, then one can not find a CP assignment map such that any arbitrary localized dynamics as Eq. (15) directly reduces to a localized subdynamics as Eq. (18). From Sec. II, we know that, for such a state, there exists, at least, one localized dynamics as Eq. (6) which its reduction is not equivalent to any localized subdynamics as Eq. (7). Now, when the initial ρ_{ABE} is not a Markov state, whether it is always possible to find a localized dynamics as Eq. (15) which its reduction is not equivalent to any localized subdynamics, remains as an open question.

IV. WHEN BOTH PARTS OF THE SYSTEM CAN INTERACT WITH THE ENVIRONMENT

Let's come back to our main subject. In Sec. II, we considered the case that the system is bipartite $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$. Then, assuming that only the party B interacts with the environment (and the party A is isolated from the environment) as Eq. (6), we asked when any such localized dynamics reduces to a localized subdynamics as Eq. (7). We have seen that this will be the case if and only if the initial ρ_{ABE} is a Markov state as Eq. (4).

Now, assume that, also, the party A can interact with the environment. So each localized dynamics as $\mathcal{F}_{SE} = id_B \otimes \mathcal{F}_{AE}$ leads to a localized subdynamics as $\mathcal{E}_{AB} = id_B \otimes \bar{\mathcal{E}}_A$ if and only if the initial ρ_{ABE} be a Markov state as

$$\begin{aligned} \rho_{ABE} &= \bigoplus_j p_j \rho_{a_j^L B} \otimes \rho_{a_j^R E}, \\ \mathcal{H}_A &= \bigoplus_j \mathcal{H}_{A_j} = \bigoplus_j \mathcal{H}_{a_j^L} \otimes \mathcal{H}_{a_j^R}, \end{aligned} \quad (24)$$

where $\{p_j\}$ is a probability distribution, $\rho_{a_j^L B}$ is a state on $\mathcal{H}_{a_j^L} \otimes \mathcal{H}_B$ and $\rho_{a_j^R E}$ is a state on $\mathcal{H}_{a_j^R} \otimes \mathcal{H}_E$.

If we require that arbitrary localized maps $id_A \otimes \mathcal{F}_{BE}$ and $id_B \otimes \mathcal{F}_{AE}$ reduce as $id_A \otimes \bar{\mathcal{E}}_B$ and $id_B \otimes \bar{\mathcal{E}}_A$, respectively, then both Eqs. (4) and (24) must be held simultaneously for the initial ρ_{ABE} . Now, consider the projection

$$\begin{aligned} \Pi_{jk} &\equiv \Pi_{A_j} \otimes \Pi_{B_k} \otimes I_E \\ &= (\Pi_{a_j^L} \otimes \Pi_{a_j^R}) \otimes (\Pi_{b_k^L} \otimes \Pi_{b_k^R}) \otimes I_E, \end{aligned} \quad (25)$$

where Π_{A_j} , Π_{B_k} , $\Pi_{a_j^L}$, $\Pi_{a_j^R}$, $\Pi_{b_k^L}$ and $\Pi_{b_k^R}$ are the projectors onto \mathcal{H}_{A_j} , \mathcal{H}_{B_k} , $\mathcal{H}_{a_j^L}$, $\mathcal{H}_{a_j^R}$, $\mathcal{H}_{b_k^L}$ and $\mathcal{H}_{b_k^R}$, respectively, and I_E is the identity operator on \mathcal{H}_E . From

Eqs. (4), (24) and (25), we have

$$\begin{aligned}\Pi_{jk}\rho_{ABE}\Pi_{jk} &= p_j \sigma_{a_j^L B_k} \otimes \rho_{a_j^R E} \\ &= q_k \sigma_{A_j b_k^L} \otimes \rho_{b_k^R E},\end{aligned}\quad (26)$$

where $\sigma_{a_j^L B_k} \equiv \Pi_{B_k} \rho_{a_j^L B} \Pi_{B_k}$ and $\sigma_{A_j b_k^L} \equiv \Pi_{A_j} \rho_{A b_k^L} \Pi_{A_j}$ are positive operators.

Note that if $\sigma_{a_j^L B_k}$ be non-zero then $\sigma_{A_j b_k^L}$ is so, and vice versa (we consider only those terms in Eqs. (4) and (24) for which $q_k \neq 0$ and $p_j \neq 0$). For each j there is at least one k for which $\sigma_{a_j^L B_k}$ and $\sigma_{A_j b_k^L}$ are non-zero. For this (j, k) , we define

$$p^{(j,k)} \equiv \text{Tr}(\sigma_{a_j^L B_k}), \quad \rho_{a_j^L B_k}^{(j,k)} \equiv \frac{\sigma_{a_j^L B_k}}{p^{(j,k)}},$$

and

$$q^{(j,k)} \equiv \text{Tr}(\sigma_{A_j b_k^L}), \quad \rho_{A_j b_k^L}^{(j,k)} \equiv \frac{\sigma_{A_j b_k^L}}{q^{(j,k)}},$$

where $\rho_{a_j^L B_k}^{(j,k)}$ is a state on $\mathcal{H}_{a_j^L} \otimes \mathcal{H}_{B_k}$ and $\rho_{A_j b_k^L}^{(j,k)}$ is a state on $\mathcal{H}_{A_j} \otimes \mathcal{H}_{b_k^L}$. So, for this (j, k) , Eq. (26) can be rewritten as

$$p_j p^{(j,k)} \rho_{a_j^L B_k}^{(j,k)} \otimes \rho_{a_j^R E} = q_k q^{(j,k)} \rho_{A_j b_k^L}^{(j,k)} \otimes \rho_{b_k^R E}.$$

By tracing from both sides, we get $p_j p^{(j,k)} = q_k q^{(j,k)}$. So

$$\rho_{a_j^L B_k}^{(j,k)} \otimes \rho_{a_j^R E} = \rho_{A_j b_k^L}^{(j,k)} \otimes \rho_{b_k^R E}. \quad (27)$$

Tracing from both sides, with respect to a_j^L and B_k , gives us

$$\rho_{a_j^R E} = \rho_{a_j^R}^{(j,k)} \otimes \bar{\rho}_E^{(k)}, \quad (28)$$

where $\rho_{a_j^R}^{(j,k)} = \text{Tr}_{a_j^L b_k^L}(\rho_{A_j b_k^L}^{(j,k)})$ is a state on $\mathcal{H}_{a_j^R}$ and $\bar{\rho}_E^{(k)} = \text{Tr}_{b_k^R}(\rho_{b_k^R E})$ is a state on \mathcal{H}_E . Similarly, by tracing from both sides of Eq. (27) with respect to A_j and b_k^L , we have

$$\rho_{b_k^R E} = \rho_{b_k^R}^{(j,k)} \otimes \hat{\rho}_E^{(j)}, \quad (29)$$

where $\rho_{b_k^R}^{(j,k)} = \text{Tr}_{a_j^L b_k^L}(\rho_{a_j^L B_k}^{(j,k)})$ is a state on $\mathcal{H}_{b_k^R}$ and $\hat{\rho}_E^{(j)} = \text{Tr}_{a_j^R}(\rho_{a_j^R E})$ is a state on \mathcal{H}_E .

Using Eq. (28), we can rewrite Eq. (24) as

$$\rho_{ABE} = \bigoplus_j p_j \rho_{a_j^L B} \otimes \rho_{a_j^R}^{(j,k)} \otimes \bar{\rho}_E^{(k)}.$$

Since k can be considered as a function of j , we can define $\rho_E^{(j)} \equiv \bar{\rho}_E^{(k(j))}$ and rewrite the above equation in a simpler form

$$\begin{aligned}\rho_{ABE} &= \bigoplus_j p_j \rho_{a_j^L B} \otimes \rho_{a_j^R} \otimes \rho_E^{(j)} \\ &= \bigoplus_j p_j \rho_{A_j B} \otimes \rho_E^{(j)},\end{aligned}\quad (30)$$

where $\rho_{A_j B} \equiv \rho_{a_j^L B} \otimes \rho_{a_j^R}$ and, in addition, we have omitted the superscript (j, k) of $\rho_{a_j^R}^{(j,k)}$.

Similarly, Using Eq. (29), we can rewrite Eq. (4) as

$$\begin{aligned}\rho_{ABE} &= \bigoplus_k q_k \rho_{A b_k^L} \otimes \rho_{b_k^R}^{(j,k)} \otimes \hat{\rho}_E^{(j)} \\ &= \bigoplus_k q_k \rho_{A b_k^L} \otimes \rho_{b_k^R} \otimes \hat{\rho}_E^{(k)} \\ &= \bigoplus_k q_k \hat{\rho}_{AB_k} \otimes \hat{\rho}_E^{(k)},\end{aligned}\quad (31)$$

where $\hat{\rho}_E^{(k)} \equiv \hat{\rho}_E^{(j(k))}$ and $\hat{\rho}_{AB_k} \equiv \rho_{A b_k^L} \otimes \rho_{b_k^R} = \rho_{A b_k^L} \otimes \rho_{b_k^R}^{(j,k)}$.

In summary, each localized dynamics, as $id_A \otimes \mathcal{F}_{BE}$ or $id_B \otimes \mathcal{F}_{AE}$, reduces to a localized subdynamics, as $id_A \otimes \mathcal{E}_B$ or $id_B \otimes \mathcal{E}_A$, respectively, if and only if both Eqs. (30) and (31) hold for the initial ρ_{ABE} . Note that in Eq. (4), A and E are separated (i.e. $\rho_{AE} = \text{Tr}_B(\rho_{ABE})$ is separable), and in Eq. (24), B and E are separated. Now, the requirement that the both Eqs. (4) and (24) must be held simultaneously, leads to Eqs. (30) and (31), which in both, E is separated from the whole $S = AB$.

Let's end this section by considering an special (maybe interesting) case. Assume that for $j = j_0$ and all k , $\sigma_{a_{j_0}^L B_k}$ and $\sigma_{A_{j_0} b_k^L}$ in Eq. (26) are non-zero. So, Eq. (28) holds for this fixed $j = j_0$ and all k :

$$\rho_{a_{j_0}^R E} = \rho_{a_{j_0}^R}^{(j_0,k)} \otimes \bar{\rho}_E^{(k)}.$$

Tracing from the both sides with respect to $a_{j_0}^R$, we get $\bar{\rho}_E^{(k)} = \text{Tr}_{a_{j_0}^R}(\rho_{a_{j_0}^R E})$, which is a fixed state for all k . So all $\rho_E^{(j)} = \bar{\rho}_E^{(k)}$ in Eq. (30) are the same (which we may denote it as ρ_E). Therefore

$$\rho_{ABE} = \left(\bigoplus_j p_j \rho_{a_j^L B} \otimes \rho_{a_j^R} \right) \otimes \rho_E;$$

i.e. the initial ρ_{ABE} is factorized.

V. WHEN EACH PART OF THE SYSTEM INTERACTS WITH ITS LOCAL ENVIRONMENT

Now, let's consider the following case which may be more interesting than the previous one. Assume that the two parties A and B of our bipartite system, are separated from each other and each one interacts with its own local environment. Let's denote the local environment of A as E_A , the local environment of B as E_B and the initial state of the system-environments as $\rho_{AE_A BE_B}$. From Sec. II, we know that if, for a $\rho_{AE_A BE_B}$, each localized

dynamics $id_{AE_A} \otimes \mathcal{F}_{BE_B}$ as

$$\begin{aligned} \rho'_{AE_A B' E'_B} &= \sum_j (I_{AE_A} \otimes f_j) \rho_{AE_A B E_B} (I_{AE_A} \otimes f_j^\dagger), \\ f_j : \mathcal{H}_B \otimes \mathcal{H}_{E_B} &\rightarrow \mathcal{H}_{B'} \otimes \mathcal{H}_{E'_B}, \\ \sum_j f_j^\dagger f_j &= I_{BE_B}, \end{aligned} \quad (32)$$

reduces to a localized subdynamics $id_{AE_A} \otimes \bar{\mathcal{E}}_B$ as

$$\begin{aligned} \rho'_{AE_A B'} &= \sum_i (I_{AE_A} \otimes \bar{E}_i) \rho_{AE_A B} (I_{AE_A} \otimes \bar{E}_i^\dagger), \\ \bar{E}_i : \mathcal{H}_B &\rightarrow \mathcal{H}_{B'}, \\ \sum_i \bar{E}_i^\dagger \bar{E}_i &= I_B, \end{aligned} \quad (33)$$

then the initial $\rho_{AE_A B E_B}$ is as

$$\begin{aligned} \rho_{AE_A B E_B} &= \bigoplus_k q_k \rho_{AE_A b_k^L} \otimes \rho_{b_k^R E_B}, \\ \mathcal{H}_B &= \bigoplus_k \mathcal{H}_{B_k} = \bigoplus_k \mathcal{H}_{b_k^L} \otimes \mathcal{H}_{b_k^R}, \end{aligned} \quad (34)$$

and vice versa. In above equations, $\rho_{AE_A B} = \text{Tr}_{E_B}(\rho_{AE_A B E_B})$, $\rho'_{AE_A B'} = \text{Tr}_{E'_B}(\rho'_{AE_A B' E'_B})$ and the final Hilbert spaces of B and E_B , $\mathcal{H}_{B'}$ and $\mathcal{H}_{E'_B}$, may differ from the initial \mathcal{H}_B and \mathcal{H}_{E_B} , respectively. In addition, $\{q_k\}$ is a probability distribution, $\rho_{AE_A b_k^L}$ is a state on $\mathcal{H}_A \otimes \mathcal{H}_{E_A} \otimes \mathcal{H}_{b_k^L}$ and $\rho_{b_k^R E_B}$ is a state on $\mathcal{H}_{b_k^R} \otimes \mathcal{H}_{E_B}$. Also note that from Eq. (33), we have

$$\begin{aligned} \rho'_{AB'} &= \text{Tr}_{E_A}(\rho'_{AE_A B'}) = \sum_i (I_A \otimes \bar{E}_i) \rho_{AB} (I_A \otimes \bar{E}_i^\dagger) \\ &= id_A \otimes \bar{\mathcal{E}}_B(\rho_{AB}), \end{aligned} \quad (35)$$

where $\rho_{AB} = \text{Tr}_{E_A}(\rho_{AE_A B})$ is the initial state of the system.

Similarly, if, for an initial $\rho_{AE_A B E_B}$, any arbitrary localized dynamics as $\rho'_{A' E'_A B E_B} = \mathcal{F}_{AE_A} \otimes id_{BE_B}(\rho_{AE_A B E_B})$ reduces to a localized subdynamics as $\rho'_{A' B E_B} = \bar{\mathcal{E}}_A \otimes id_{BE_B}(\rho_{AB E_B})$, where $\rho_{AB E_B} = \text{Tr}_{E_A}(\rho_{AE_A B E_B})$ and $\rho'_{A' B E_B} = \text{Tr}_{E'_A}(\rho'_{A' E'_A B E_B})$, then

$$\begin{aligned} \rho_{AE_A B E_B} &= \bigoplus_j p_j \rho_{a_j^L E_A} \otimes \rho_{a_j^R B E_B}, \\ \mathcal{H}_A &= \bigoplus_j \mathcal{H}_{A_j} = \bigoplus_j \mathcal{H}_{a_j^L} \otimes \mathcal{H}_{a_j^R}, \end{aligned} \quad (36)$$

and vice versa. In the above equation, $\{p_j\}$ is a probability distribution, $\rho_{a_j^L E_A}$ is a state on $\mathcal{H}_{a_j^L} \otimes \mathcal{H}_{E_A}$ and $\rho_{a_j^R B E_B}$ is a state on $\mathcal{H}_{a_j^R} \otimes \mathcal{H}_B \otimes \mathcal{H}_{E_B}$.

Now, if, for an initial $\rho_{AE_A B E_B}$, each localized dynamics as $id_{AE_A} \otimes \mathcal{F}_{BE_B}$ or $\mathcal{F}_{AE_A} \otimes id_{BE_B}$ reduces to a localized subdynamics as $id_{AE_A} \otimes \bar{\mathcal{E}}_B$ or $\bar{\mathcal{E}}_A \otimes id_{BE_B}$, respectively, then both Eqs. (34) and (36) hold simultaneously (and vice versa).

Define the projection

$$\Pi_k = \Pi_{B_k} \otimes I_{AE_A E_B}, \quad (37)$$

where Π_{B_k} is the projection onto \mathcal{H}_{B_k} and $I_{AE_A E_B}$ is the identity operator on $\mathcal{H}_A \otimes \mathcal{H}_{E_A} \otimes \mathcal{H}_{E_B}$. So, using Eqs. (34), (36) and (37), we have

$$\begin{aligned} \Pi_k \rho_{AE_A B E_B} \Pi_k &= q_k \rho_{AE_A b_k^L} \otimes \rho_{b_k^R E_B} \\ &= \bigoplus_j p_j \rho_{a_j^L E_A} \otimes \sigma_{a_j^R B_k E_B}, \end{aligned} \quad (38)$$

where $\sigma_{a_j^R B_k E_B} = \bar{\Pi}_{jk} \rho_{a_j^R B E_B} \bar{\Pi}_{jk}$ and $\bar{\Pi}_{jk} \equiv \Pi_{a_j^R} \otimes \Pi_{B_k} \otimes I_{E_B}$ (where $\Pi_{a_j^R}$ is the projection onto $\mathcal{H}_{a_j^R}$). $\sigma_{a_j^R B_k E_B}$ is a positive operator on $\mathcal{H}_{a_j^R} \otimes \mathcal{H}_{B_k} \otimes \mathcal{H}_{E_B}$. Let $p'_{jk} = \text{Tr}(\sigma_{a_j^R B_k E_B})$; so $0 \leq p'_{jk} \leq 1$. Now if $p'_{jk} > 0$, we define

$$\rho_{a_j^R B_k E_B} = \frac{\sigma_{a_j^R B_k E_B}}{p'_{jk}},$$

otherwise, if $p'_{jk} = 0$, we define $\rho_{a_j^R B_k E_B}$ arbitrarily. So, Eq. (38) can be rewritten as

$$q_k \rho_{AE_A b_k^L} \otimes \rho_{b_k^R E_B} = \bigoplus_j p_j p'_{jk} \rho_{a_j^L E_A} \otimes \rho_{a_j^R B_k E_B}.$$

Tracing from both sides, with respect to b_k^R and E_B , we get

$$\rho_{AE_A b_k^L} = \bigoplus_j p_j p_{jk} \rho_{a_j^L E_A} \otimes \rho_{a_j^R b_k^L}, \quad (39)$$

where $\rho_{a_j^R b_k^L} = \text{Tr}_{b_k^R E_B}(\rho_{a_j^R B_k E_B})$ and $p_{jk} = \frac{p_j p'_{jk}}{q_k}$ (note that we only consider those terms in Eqs. (34) and (36) for which $q_k \neq 0$ and $p_j \neq 0$). So we can rewrite Eqs. (34) as

$$\begin{aligned} \rho_{AE_A B E_B} &= \bigoplus_{jk} q_k p_{jk} \rho_{a_j^L E_A} \otimes \rho_{a_j^R b_k^L} \otimes \rho_{b_k^R E_B} \\ &= \bigoplus_{jk} q_{jk} \rho_{a_j^L E_A} \otimes \rho_{a_j^R b_k^L} \otimes \rho_{b_k^R E_B}, \\ \mathcal{H}_A &= \bigoplus_j \mathcal{H}_{a_j^L} \otimes \mathcal{H}_{a_j^R}, \mathcal{H}_B = \bigoplus_k \mathcal{H}_{b_k^L} \otimes \mathcal{H}_{b_k^R}, \end{aligned} \quad (40)$$

where $q_{jk} = q_k p_{jk}$. Note that $\rho_{AE_B} = \text{Tr}_{E_A}(\rho_{AE_A B E_B})$, $\rho_{BE_A} = \text{Tr}_{E_B}(\rho_{AE_A B E_B})$ and $\rho_{EA E_B} = \text{Tr}_{AB}(\rho_{AE_A B E_B})$ are all separable states, but ρ_{AB} may be entangled.

In addition,

$$\begin{aligned} \rho_{AB} &= \text{Tr}_{E_A E_B}(\rho_{AE_A B E_B}) \\ &= \bigoplus_{jk} q_{jk} \rho_{a_j^L} \otimes \rho_{a_j^R b_k^L} \otimes \rho_{b_k^R}, \end{aligned} \quad (41)$$

where $\rho_{a_j^L} = \text{Tr}_{E_A}(\rho_{a_j^L E_A})$ and $\rho_{b_k^R} = \text{Tr}_{E_B}(\rho_{b_k^R E_B})$. So, if we define the CP assignment maps $\Lambda_{a_j^L} : \mathcal{L}(\mathcal{H}_{a_j^L}) \rightarrow \mathcal{L}(\mathcal{H}_{a_j^L} \otimes \mathcal{H}_{E_A})$ and $\Lambda_{b_k^R} : \mathcal{L}(\mathcal{H}_{b_k^R}) \rightarrow \mathcal{L}(\mathcal{H}_{b_k^R} \otimes \mathcal{H}_{E_B})$ as

$$\begin{aligned}\Lambda_{a_j^L}(\rho_{a_j^L}) &= \rho_{a_j^L E_A}, \\ \Lambda_{b_k^R}(\rho_{b_k^R}) &= \rho_{b_k^R E_B},\end{aligned}\quad (42)$$

then we can define the CP assignment map $\Lambda : \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_{E_A} \otimes \mathcal{H}_B \otimes \mathcal{H}_{E_B})$ as $\rho_{AE_A BE_B} = \Lambda(\rho_{AB}) = \Lambda_A \otimes \Lambda_B(\rho_{AB})$, where $\Lambda_A = \bigoplus_j \Lambda_{a_j^L} \otimes id_{a_j^R}$ and $\Lambda_B = \bigoplus_k id_{b_k^L} \otimes \Lambda_{b_k^R}$ are CP assignment maps on $\mathcal{L}(\mathcal{H}_A)$ and $\mathcal{L}(\mathcal{H}_B)$, respectively ($id_{a_j^R}$ and $id_{b_k^L}$ are the identity maps on $\mathcal{L}(\mathcal{H}_{a_j^R})$ and $\mathcal{L}(\mathcal{H}_{b_k^L})$, respectively).

Therefore, for each localized map as $\mathcal{F}_{AE_A} \otimes \mathcal{F}_{BE_B}$, where \mathcal{F}_{AE_A} (\mathcal{F}_{BE_B}) is a CP map on $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_{E_A})$ ($\mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_{E_B})$), we have

$$\begin{aligned}\rho'_{AB} &= \text{Tr}_{E_A E_B}(\rho'_{AE_A BE_B}) \\ &= \text{Tr}_{E_A E_B}[\mathcal{F}_{AE_A} \otimes \mathcal{F}_{BE_B}(\rho_{AE_A BE_B})] \\ &= \text{Tr}_{E_A E_B}[(\mathcal{F}_{AE_A} \otimes \mathcal{F}_{BE_B})(\Lambda_A \otimes \Lambda_B)(\rho_{AB})] \\ &= (\text{Tr}_{E_A} \circ \mathcal{F}_{AE_A} \circ \Lambda_A) \otimes (\text{Tr}_{E_B} \circ \mathcal{F}_{BE_B} \circ \Lambda_B)(\rho_{AB}) \\ &= \bar{\mathcal{E}}_A \otimes \bar{\mathcal{E}}_B(\rho_{AB}),\end{aligned}\quad (43)$$

where $\bar{\mathcal{E}}_A \equiv \text{Tr}_{E_A} \circ \mathcal{F}_{AE_A} \circ \Lambda_A$ and $\bar{\mathcal{E}}_B \equiv \text{Tr}_{E_B} \circ \mathcal{F}_{BE_B} \circ \Lambda_B$ are CP maps on $\mathcal{L}(\mathcal{H}_A)$ and $\mathcal{L}(\mathcal{H}_B)$, respectively.

In summary, if the initial $\rho_{AE_A BE_B}$ be as Eq. (40), then each localized dynamics as $\mathcal{F}_{AE_A} \otimes \mathcal{F}_{BE_B}$ reduces to a localized subdynamics as $\bar{\mathcal{E}}_A \otimes \bar{\mathcal{E}}_B$. In addition, if each localized dynamics as $id_{AE_A} \otimes \mathcal{F}_{BE_B}$ or $\mathcal{F}_{AE_A} \otimes id_{BE_B}$ reduces to a localized subdynamics as $id_{AE_A} \otimes \bar{\mathcal{E}}_B$ or $\bar{\mathcal{E}}_A \otimes id_{BE_B}$, respectively, then the initial $\rho_{AE_A BE_B}$ is given by Eq. (40) (and vice versa).

VI. CONCLUSION

We considered a bipartite quantum system including parties A and B . Assuming that the dynamics of the system-environment is given by $id_A \otimes \mathcal{F}_{BE}$, we questioned whether the reduced dynamics of the system is as $id_A \otimes \bar{\mathcal{E}}_B$. At the first look, one may expect that this will be the case, since when the dynamics is given by $id_A \otimes \mathcal{F}_{BE}$, it means that the part A is isolated from the environment and its reduced state remains unchanged during the evolution. But, as we saw in Sec. II, only for Markov states, Eq. (4), each dynamics as $id_A \otimes \mathcal{F}_{BE}$ reduces to a subdynamics as $id_A \otimes \bar{\mathcal{E}}_B$. In addition, we gave some illustrating examples in that section, too.

In Secs. III, we proved that the initial ρ_{ABE} is a Markov state if and only if there exists a CP assignment map Λ such that each localized dynamics as Eq. (15) *directly* reduces to a localized subdynamics. When the initial ρ_{ABE} is not a Markov state, then there is no such Λ and one can find, at least, one localized dynamics which does not reduce to a localized subdynamics. Whether this localized dynamics is in the form of Eq. (15) or in the (more general) form of Eq. (6), remained as an open question.

In Secs. IV and V, we generalized the result given in Sec. II. When the both parts of the system, A and B , can interact with the environment was considered in Sec. IV and when each part of the system interacts with its local environment was discussed in Sec. V. For example, in Sec. V, we have shown that each localized dynamics as $\mathcal{F}_{AE_A} \otimes \mathcal{F}_{BE_B}$, where $E_A(E_B)$ is the local environment of A (B), reduces to a localized subdynamics as $\bar{\mathcal{E}}_A \otimes \bar{\mathcal{E}}_B$, if the initial state of the system and its environments is given by Eq. (40).

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